Year 11 Mathematics Extension I Term 4 2009

Question 1 **Marks**

a) (i) Find $\int xe^{x^2}\,dx$. **1**

(ii) Find $\int \frac{x+1}{x+2}\,dx$. **2**

b) Use Simpson's rule with 3 function values to estimate $\int_1^3 x\ln x \; dx$ correct to 2 decimal places. **2**

c) (i) Express $\sqrt{3}\sin x - \cos x$ in the form of $R\sin(x-\alpha)$ where $R>0$ and $0 \le \alpha \le 2\pi$. **2**

(ii) Find the minimum of $\sqrt{3}\sin x - \cos x$. **1**

(iii) Find the first positive value of x for which $\sqrt{3}\sin x - \cos x$ is a minimum. **1**

Question 2 (Start a new page)

a) The points $A(6,5)$, $B(2,0)$ and $C\,(8,3)$ are the vertices of a triangle. Calculate the length of the altitude through A. **3**

b) Find $\lim_{x\to 0}(\sin 3x \div \tan\frac{x}{3})$. **3**

c) Find the area enclosed by the curve of $y = \frac{1}{\sqrt{16-x^2}}$, the x-axis and $x = 2$, $x = -2$. **3**

Question 3 (Start a new page)

a) Find $\int \sqrt{3x-1}\,dx$. **1**

b) Consider the function $f(x) = \sqrt{4-\sqrt{x}}$.

(i) Explain why the domain of $f(x)$ is $0 \le x \le 16$. **1**

(ii) Prove that $f(x)$ is a decreasing function. **1**

(iii) Find the range of $f(x)$. **1**

(iv) Find the equation of $f^{-1}(x)$. **2**

(v) By considering the graphs of $f(x)$ and $f^{-1}(x)$, prove that **3**

$$\int_0^{16} \sqrt{4-\sqrt{x}} \; dx = 17\frac{1}{15}.$$

Question 4 (Start a new page) **Marks**

a) Differentiate $y = \cos^{-1}(x^2)$. **1**

b) Given that $\sin^{-1} x$ and $\cos^{-1} x$ are acute,

(i) Show that $\sin(\sin^{-1} x - \cos^{-1} x) = 2x^2 - 1$. **2**

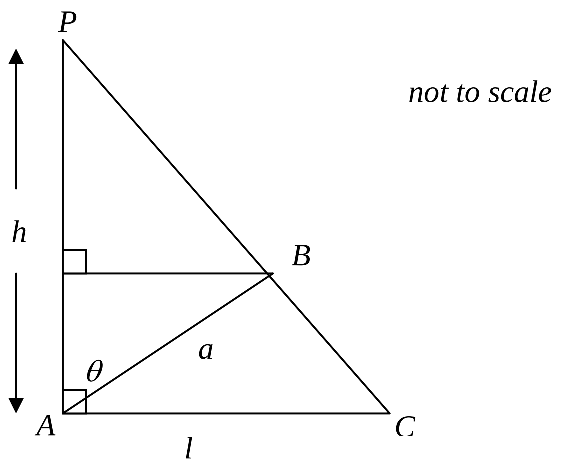
(ii) Solve the equation $\sin^{-1} x - \cos^{-1} x = \sin^{-1}(0.5)$. **2**

c) The region bounded by the curve $xy = 3$, the lines $x = 1$ and $y = 6$, and the axes is rotated about the y-axis.
Find the volume of the solid formed. **4**

Question 5 (Start a new page)

a) Use the substitution $u = e^x$ to find $\int e^{e^x + x}\,dx$. **2**

b) A rod AB of length a is hinged to a horizontal table at A. The rod is inclined to the vertical at an angle θ. There is a light located at point P at a height h vertically above A. AC of length l is the shadow of the rod on the table.



(i) Prove that $l = \dfrac{ah\sin\theta}{h - a\cos\theta}$. (You may assume similar triangles without proof) **2**

(ii) Prove that as θ varies, the maximum length of the shadow is $\dfrac{ah}{\sqrt{h^2 - a^2}}$. **5**

Question 6 (Start a new page) **Marks**

a) Use the substitution $u = \sqrt{x}$ to evaluate $\int_1^3 \frac{dx}{\sqrt{x} + x\sqrt{x}}$. **3**

b) (i) Sketch the graph $y = 2\cos^{-1}\left(\frac{x}{3}\right)$. **2**

(ii) Find the area of the region enclosed by between the graph $y = 2\cos^{-1}\left(\frac{x}{3}\right)$, the axes and $x = \frac{3}{2}$. **4**

Question 7 (Start a new page)

a) Find the exact value of $\tan^{-1}(\tan\frac{3\pi}{4})$. **1**

b) Ming was asked to sketch the graph of $y = \frac{x}{\ln(x^2)}$.

(i) He was thinking of changing $\ln(x^2)$ to $2\ln(x)$. Does this change the graph? Why? **1**

(ii) Prove that $y = \frac{x}{\ln(x^2)}$ is an odd function. **1**

(iii) Find the turning point(s) of $y = \frac{x}{\ln(x^2)}$ and determine the nature. **4**

(iv) Sketch the graph of $y = \frac{x}{\ln(x^2)}$, showing all the important features. **2**

End of Paper

Q1.

a i) $\frac{1}{2}e^{x^2}+c$ # 1

ii) $\int 1-\frac{1}{x+2}\,dx = x-\ln(x+2)+c$ 1 1 #

b) $\frac{3-1}{6}(1\ln 1+4\cdot 2\ln 2+3\ln 3)$ 1

$= \frac{1}{3}(8\ln 2+3\ln 3)$

$= 2.947\cdots$

$= 2.95$ (2dp) # 1

c i) $\sqrt{3}\sin x-\cos x = R\sin x\cos\alpha - R\cos x\sin\alpha$

$\sqrt{3} = R\cos\alpha \qquad 1 = R\sin\alpha$

$R=\sqrt{3+1}=2 \quad (R>0)$ 1

$\cos\alpha = \sqrt{3}/2$ and $\sin\alpha=\frac{1}{2}$

α is in Quad I

$\therefore \alpha = \frac{\pi}{6}$ 1

$2\sin(x-\frac{\pi}{6}) = \sqrt{3}\sin x-\cos x$

1

ii) min $=-2$

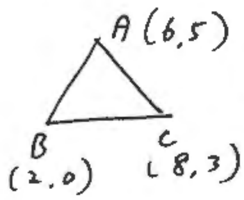
iii) $\sin(x-\frac{\pi}{6}) = -1 = \sin(\frac{3\pi}{2})$

$x-\frac{\pi}{6} = \frac{3\pi}{2}$

$x = \frac{10\pi}{6} = \frac{5\pi}{3}$ # 1

Q 2

a) Equation of BC:



$\frac{y-0}{x-2} = \frac{3-0}{8-2} = \frac{1}{2}$

$y=\frac{1}{2}(x-2)$

$2y-x+2=0$

$\therefore -x+2y+2=0$ 1

altitude through A : $\left|\frac{6(-1)+5(2)+2}{\sqrt{5}}\right|$ 1

$=\frac{6}{\sqrt{5}}$ # 1

2b) $\lim_{x\to 0}\left(\frac{\sin 3x}{3x}\right)\left(\frac{\frac{x}{3}}{\tan\frac{x}{3}}\right)\cdot 9$ 1

$= 1\times 1\times 9$ 1

$= 9$ # 1

2c) Area $= \int_{-2}^{2}\frac{dx}{\sqrt{16-x^2}}$ 1

$= 2\int_0^2 \frac{dx}{\sqrt{16-x^2}}$ (Even function)

$= 2\left[\sin^{-1}\left(\frac{x}{4}\right)\right]_0^2$ 1

$= 2\left[\sin^{-1}\left(\frac{1}{2}\right)-0\right]$

$= \frac{\pi}{3}$ # 1

Q 3

a) $\frac{\frac{2}{3}(3x-1)^{3/2}}{3}+c$

$= \frac{2}{9}(3x-1)^{3/2}+c$ # 1

b i) $\sqrt{4-\sqrt{x}} \geq 0$

$4-\sqrt{x}\geq 0$

$4\geq\sqrt{x}$, $x\geq 0$

$\therefore 16\geq x\geq 0$ 1

ii) $f'(x) = \frac{1}{2\sqrt{4-\sqrt{x}}}\cdot\frac{-1}{2\sqrt{x}}$

$f'(x) = \frac{-1}{4\sqrt{x}\sqrt{4-\sqrt{x}}} < 0$ 1

$\therefore f(x)$ is a decreasing function #

Q 4

(iii) $f(x)=\sqrt{4-\sqrt{x}}$

$0 \le x \le 16$

when $x=0$, $f(x)=2$

$x=16$ $\quad f(x)=0$

but $f(x)\ge 0$

$\therefore \; 0\le f(x)\le 2$ 1

iv) Given $y=\sqrt{4-\sqrt{x}}$ $\quad 0\le x\le 16$, $0\le y\le 2$

To find $f^{-1}(x)$:

$x=\sqrt{4-\sqrt{y}}$ 1

$x^2=4-\sqrt{y}$

$\sqrt{y}=4-x^2$

$f^{-1}(x)=y=(4-x^2)^2, \quad 0\le x\le 2$ 1

v) $\int_0^{16}\sqrt{4-\sqrt{x}}\,dx=\int_0^2(4-x^2)^2\,dx$ 1

$=\int_0^2 16-8x^2+x^4\,dx$

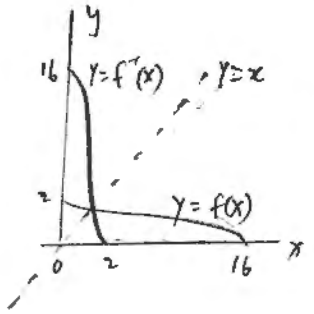
$=\left[16x-\frac{8}{3}x^3+\frac{x^5}{5}\right]_0^2$ 1

$=32-\frac{64}{3}+\frac{32}{5}$ 1

$=\frac{480-320+96}{15}$

$=\frac{256}{15}$

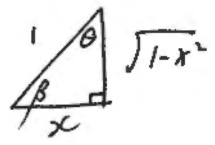
$=17\frac{1}{15}$ #



a) $y'=\frac{-1}{\sqrt{1-x^4}}\cdot 2x=\frac{-2x}{\sqrt{1-x^4}}$ # 1

b(i) Let $\theta=\sin^{-1}x$

$\beta=\cos^{-1}x$



$\sin(\sin^{-1}x-\cos^{-1}x)$

$=\sin(\theta-\beta)=\sin\theta\cos\beta-\cos\theta\sin\beta$

$=x\cdot x-(\sqrt{1-x^2})(\sqrt{1-x^2})$ 1

$=x^2-(1-x^2)$ 1

$=2x^2-1$ #

ii) Solve $\sin^{-1}x-\cos^{-1}x=\sin^{-1}(0.5)$

$\sin(\sin^{-1}x-\cos^{-1}x)=\sin(\sin^{-1}(0.5))$ $\frac{1}{2}$

Fr. (i) $2x^2-1=0.5$ $\frac{1}{2}$

$2x^2=1.5$

$x^2=0.75=\frac{3}{4}$

$x=\pm\frac{\sqrt{3}}{2}$ $\frac{1}{2}$

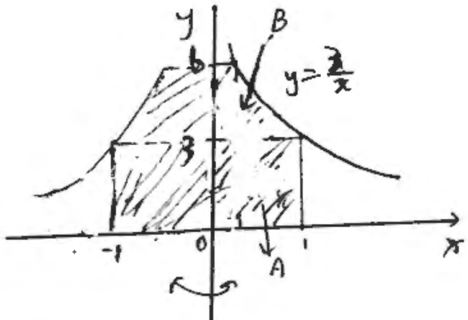
but θ, β are acute

$1\ge x\ge 0$

$\therefore x=\frac{\sqrt{3}}{2}$ only $\frac{1}{2}$

#

c)



$x=1, y=3$ $\frac{1}{2}$

vol (A) = vol of cylinder

$=(\pi\cdot 1^2)3=3\pi$ 1

vol B $=\int_3^6\pi x^2\,dy=\pi\int_3^6\left(\frac{3}{y}\right)^2dy$ 1

$=9\pi\int_3^6\frac{dy}{y^2}=9\pi\left[-\frac{1}{y}\right]_3^6=-\frac{9\pi}{6}+3\pi$ $\frac{1}{2}$

$=\frac{3\pi}{2}$ $\frac{1}{2}$

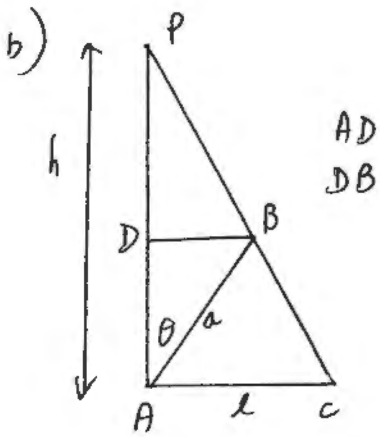
Total vol $=3\pi+\frac{3\pi}{2}=\frac{9\pi}{2}$ unit^3 # $\frac{1}{2}$

a) $u = e^x \quad du = e^x dx$

$$\int e^{e^x} \cdot e^x dx = \int e^u du \qquad 1$$

$$= e^u + c = e^{e^x} + c \quad \# \qquad 1$$

b)



$AD = a\cos\theta$

$DB = a\sin\theta$

(i) $\triangle PBD \;|||\; \triangle PAC$

$$\therefore \frac{DB}{AC} = \frac{PD}{PA} \qquad 1$$

$$\frac{a\sin\theta}{AC} = \frac{h - a\cos\theta}{h}$$

$$\frac{a\sin\theta}{l} = \frac{h - a\cos\theta}{h} \qquad 1$$

$$l = \frac{ah\sin\theta}{h - a\cos\theta} \quad \#$$

(ii) $l = \dfrac{ah\sin\theta}{h - a\cos\theta}$

$$\frac{dl}{d\theta} = \frac{(h - a\cos\theta)ah\cos\theta - (ah\sin\theta)a\sin\theta}{(h - a\cos\theta)^2} \qquad 1$$

$\frac{dl}{d\theta} = 0$

$$\therefore ah^2\cos\theta \overbrace{- a^2h\cos^2\theta - a^2h\sin^2\theta}^{-a^2h} = 0$$

$$ah^2\cos\theta - a^2h = 0$$

$$h\cos\theta = a$$

$$\cos\theta = \frac{a}{h} \quad \# \qquad 1$$

Since $\cos\theta$ is a decreasing function for $0 \le \theta \le \frac{\pi}{2}$ and limit $\varepsilon \to 0^+$

$$\cos(\theta + \varepsilon) > \frac{a}{h}$$

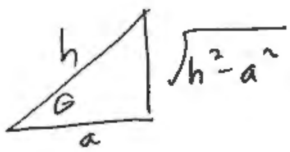
$$\cos(\theta - \varepsilon) < \frac{a}{h}$$

θ	$\cos^{-1}\left(\frac{a}{h}\right) - \varepsilon$	$\cos^{-1}\left(\frac{a}{h}\right)$	$\cos^{-1}\left(\frac{a}{h}\right) + \varepsilon$
$\frac{dl}{d\theta}$	+ve	0	−ve
	/	—	\

1

$\therefore \theta = \cos^{-1}\left(\frac{a}{h}\right)$ is rel max

Since l is continuous for $0 \le \theta \le \frac{\pi}{2}$ and no other T.P $\theta = \cos^{-1}\left(\frac{a}{h}\right)$ give absolute max $\frac{1}{2}$



$\sin\theta = \dfrac{\sqrt{h^2 - a^2}}{h} \quad \frac{1}{2}$

$\cos\theta = \dfrac{a}{h}$

$$l = \frac{ah\sin\theta}{h - a\cos\theta} = \frac{a\sqrt{h^2 - a^2}}{h - \frac{a^2}{h}}$$

$$l = \frac{a\sqrt{h^2 - a^2}\cdot h}{h^2 - a^2} = \frac{ah}{\sqrt{a^2 - h^2}} \quad \#$$

1

Q6

a) $u = \sqrt{x}$

$du = \frac{dx}{2\sqrt{x}}$

$2u\,du = dx$

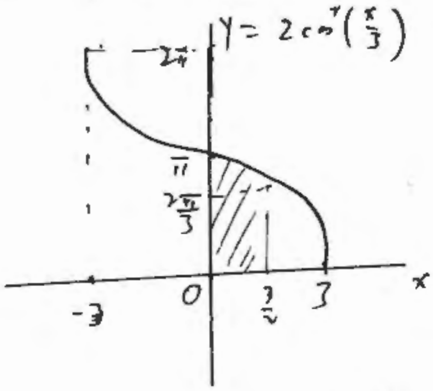
$x=1, \quad u=1$

$x=3 \quad u=\sqrt{3}$

1 $\quad \int_1^{\sqrt{3}} \frac{2u\,du}{u(1+u^2)} = 2\int_1^{\sqrt{3}} \frac{du}{1+u^2} = 2\left[\tan^{-1} u\right]_1^{\sqrt{3}}$ 1

$= 2\left[\frac{\pi}{3} - \frac{\pi}{4}\right] = \frac{\pi}{6}$ 1

b:)



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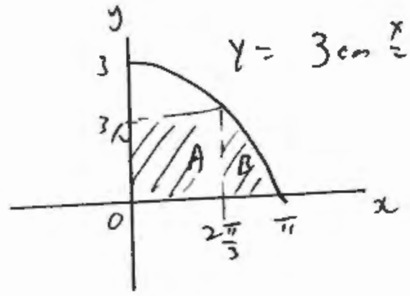
$x = \frac{3}{2}, \quad y = \frac{2\pi}{3}$ $\quad \frac{1}{2}$



Part A is a rectangle

Area $A = \frac{2\pi}{3}\cdot\frac{3}{2} = \pi$ 1

Area Part $B = \int_{\frac{2\pi}{3}}^{\pi} 3\cos\frac{x}{2}\,dx$ 1

$= 3\cdot 2\left(\sin\frac{x}{2}\right)_{\frac{2\pi}{3}}^{\pi}$

$= 6 - 6\frac{\sqrt{3}}{2} = 6 - 3\sqrt{3}$ 1

Total $(A+B) = \pi + 6 - 3\sqrt{3}$ unit^2 # $\frac{1}{2}$

Q7

a) $\frac{\pi}{4}$ 1

b:) Yes, lost left half of the graph.

FOR $y = \frac{x}{\ln x^2}$ D: $x \in R$ but $x \neq 0, \pm$

FOR $y = \frac{x}{2\ln x}$ D: $x > 0$ but $x \neq 1$ 1

ii) $f(-x) = \frac{-x}{\ln(-x)^2} = \frac{-x}{\ln x^2} = -f(x)$ 1

$\therefore$ odd

iii) Since $f(x)$ is odd, only need to consider values of $x > 0$ and use point symmetry.

$y = \frac{x}{\ln x^2} = \frac{x}{2\ln x} \qquad x>0, x \neq 1$

$y' = \frac{2\ln x - x\cdot\frac{2}{x}}{(2\ln x)^2} = \frac{2(\ln x - 1)}{4(\ln x)^2}$ 1

$y' = \frac{\ln x - 1}{2(\ln x)^2}$

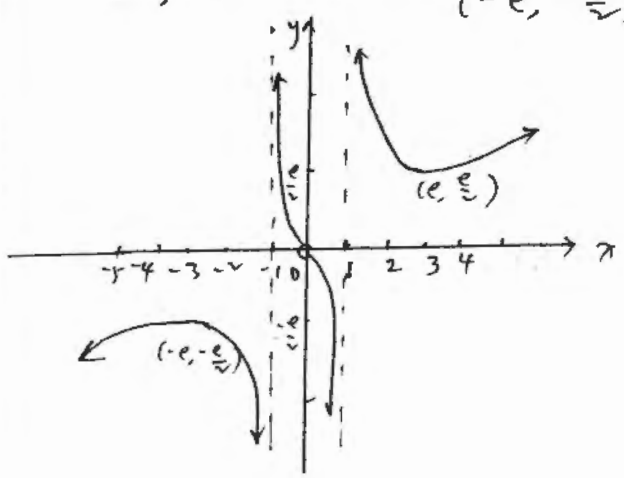
SP $y' = 0$ when $x = e$, $y = \frac{e}{2}$ 1

check max / min

x	2	e	3
y'	-0.32	0	0.04

local min at $\left(e, \frac{e}{2}\right)$ 1

using point symmetry local max at $\left(-e, -\frac{e}{2}\right)$ 1



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